

AN AFFINE SPHERE EQUATION ASSOCIATED TO EINSTEIN TORIC SURFACES

TOSHIKI MABUCHI

ABSTRACT. As seen in the works of Calabi [1], Cheng-Yau [3] and Loftin [8], affine sphere equations have a close relationship with Kähler-Einstein metrics. The main purpose of this note is to show that an equation analogous to those of hyperbolic affine spheres arises naturally from Kähler-Einstein metrics on Einstein toric surfaces. The case for the remaining toric surfaces with Kähler-Ricci solitons will also be discussed.

1. INTRODUCTION

In this note, we consider strictly convex bounded C^2 -domains Ω_i , $i = 1, 2, 3$, in $\mathbb{R}^2 := \{(s, t)\}$ defined as the intersections

$$\Omega_1 := \cap_{i=1}^3 \Omega_{i,1}, \quad \Omega_2 := \cap_{i=1}^4 \Omega_{i,2}, \quad \Omega_3 := \cap_{i=1}^6 \Omega_{i,3},$$

where $\Omega_{i,j}$ are the open subsets $\{(s, t) \in \mathbb{R}^2 \mid \rho_{ij}(s, t) > 0\}$ of \mathbb{R}^2 with the functions $\rho_{ij} = \rho_{ij}(s, t)$ defined by

$$\left\{ \begin{array}{l} \rho_{11} := \frac{5}{8} - \frac{3}{2}(s-t)^2 - \frac{1}{2}(s+t), \\ \rho_{21} := s - \frac{3}{2}\left(t + \frac{1}{6}\right)^2 + \frac{2}{3}, \\ \rho_{31} := t - \frac{3}{2}\left(s + \frac{1}{6}\right)^2 + \frac{2}{3}. \end{array} \right.$$

$$\left\{ \begin{array}{l} \rho_{12} := -t - \frac{3}{2}s^2 + \frac{1}{2}, \\ \rho_{22} := -s - \frac{3}{2}t^2 + \frac{1}{2}, \\ \rho_{32} := t - \frac{3}{2}s^2 + \frac{1}{2}, \\ \rho_{42} := s - \frac{3}{2}t^2 + \frac{1}{2}, \end{array} \right.$$

$$\left\{ \begin{array}{l} \rho_{13} := -t - \frac{3}{2} \left(s - t + \frac{1}{6}\right)^2 + \frac{1}{3}, \\ \rho_{23} := -s - \frac{3}{2} \left(t - \frac{1}{6}\right)^2 + \frac{1}{3}, \\ \rho_{33} := t - \frac{3}{2} \left(s + \frac{1}{6}\right)^2 + \frac{1}{3}, \\ \rho_{43} := t - \frac{3}{2} \left(-s + t + \frac{1}{6}\right)^2 + \frac{1}{3}, \\ \rho_{53} := s - \frac{3}{2} \left(t + \frac{1}{6}\right)^2 + \frac{1}{3}, \\ \rho_{63} := -t - \frac{3}{2} \left(s - \frac{1}{6}\right)^2 + \frac{1}{3}. \end{array} \right.$$

Then each boundary curve $\partial\Omega_j := \bar{\Omega}_j \setminus \Omega_j$ in \mathbb{R}^2 is not only C^2 but also piecewise quadratic. By a theorem of Cheng and Yau [2], there exists a unique convex negative solution $\varphi = \varphi(s, t) \in C^\infty(\Omega_j) \cap C^0(\bar{\Omega}_j)$ for

$$(1.1) \quad \left\{ \begin{array}{l} (-\varphi)^{2+k} \det \text{Hess}(\varphi) = 1 \quad \text{on } \Omega_j, \\ \varphi|_{\partial\Omega_j} = 0, \end{array} \right.$$

where the convex negativity of the solution φ for (1.1) means that $\varphi < 0$ on Ω_j and that the Hessian matrix

$$\text{Hess}(\varphi) := \begin{pmatrix} \varphi_{ss} & \varphi_{st} \\ \varphi_{st} & \varphi_{tt} \end{pmatrix}$$

is positive definite everywhere on Ω_j . Then for $k = 2$, the equation (1.1) is known as the equation for hyperbolic affine spheres. In this note, we assume $k = 1/2$, and consider the solution φ of (1.1). Then by setting

$$(1.2) \quad \psi := - \left(\frac{2}{3}\right)^{\frac{2}{3}} (-\varphi)^{\frac{3}{2}} \in C^\infty(\Omega_j) \cap C^0(\bar{\Omega}_j),$$

we can rewrite (1.1) in the form

$$(1.3) \quad \left\{ \begin{array}{l} \begin{vmatrix} \psi_{ss} & \psi_{st} & \psi_s \\ \psi_{st} & \psi_{tt} & \psi_t \\ \psi_s & \psi_t & 3\psi \end{vmatrix} = -3 \quad \text{on } \Omega_j, \\ \psi|_{\partial\Omega_j} = 0. \end{array} \right.$$

In a neighborhood of $\partial\Omega_j$ in \mathbb{R}^2 , we fix a C^2 -function ρ defining $\partial\Omega_j$ such that the 1-form $d\rho$ coincides with $d\rho_{ij}$ when restricted to $\partial\Omega_j \cap \{\rho_{ij} = 0\}$ for all i . Then ψ is expressible as $-\rho - f\rho^2 +$

higher order terms in ρ . By abuse of terminology, we call the restriction

$$P(\Omega_j) := f|_{\partial\Omega_j}$$

the “Fubini-Pick invariant” of the domain Ω_j (cf. [11]). We now put $X_1 := \mathbb{P}^2(\mathbb{C})$, $X_2 := \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, $X_3 := \mathbb{P}^2(\mathbb{C}) \# 3\bar{\mathbb{P}}^2(\mathbb{C})$. Since X_j , $j = 1, 2, 3$, are toric surfaces, we have natural torus embeddings

$$T := (\mathbb{C}^*)^2 \hookrightarrow X_j.$$

In this note, by setting $k := 1/2$, we shall show that the equation for Kähler-Einstein metrics on X_j , $j = 1, 2, 3$, has a reduction to (1.1) above, where $P(\Omega_j)$ is uniquely determined by the pullback to $X_j \setminus T$ of the Kähler-Einstein form on X_j . Moreover, from the data $P(\omega_j)$, we can explicitly describe the Kähler-Einstein metric on X_j .

2. REDUCTION TO (1.1)

For toric surfaces X_j in the introduction, we consider a K -invariant Kähler-Einstein form ω on X_j in the class $2\pi c_1(X_j)$ (cf. [12], [13], [14]), where $K := S^1 \times S^1$ denotes the maximal compact subgroup of the algebraic torus $T = \mathbb{C}^* \times \mathbb{C}^*$. In view of the torus embedding

$$T = \{(z_1, z_2) \in \mathbb{C}^* \times \mathbb{C}^*\} \hookrightarrow X_j,$$

we can regard (z_1, z_2) as a system of holomorphic local coordinates on the Zariski open dense subset T of X_j . Then the restriction to T of the volume form ω^2 on X_j is written as

$$\omega^2|_T = 2e^{-h} \left(\sqrt{-1} \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} \right) \wedge \left(\sqrt{-1} \frac{dz_2 \wedge d\bar{z}_2}{|z_2|^2} \right)$$

for some K -invariant function $h \in C^\infty(T)_\mathbb{R}$ on T . Define K -invariant functions $x, y \in C^\infty(T)_\mathbb{R}$ on T by

$$e^x = |z_1|^2 \quad \text{and} \quad e^y = |z_2|^2,$$

and these are seen as real-valued independent variables with ranges $-\infty < x < +\infty$ and $-\infty < y < +\infty$. In particular, h is regarded as a smooth function

$$h = h(x, y) \in C^\infty(\mathbb{R}^2)$$

on $\mathbb{R}^2 = \{(x, y)\}$. By setting $\text{Hess}(h) := \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix}$, we see that

$$\text{Ric}(\omega)^2|_T = 2 \det \text{Hess}(h) \left(\sqrt{-1} \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} \right) \wedge \left(\sqrt{-1} \frac{dz_2 \wedge d\bar{z}_2}{|z_2|^2} \right).$$

Since ω is a Kähler-Einstein form, we have $\text{Ric}(\omega) = \omega$, and hence

$$(2.1) \quad \det \text{Hess}(h) = e^{-h}.$$

Let $\square_j := \bar{\square}_j \setminus \partial \square_j$ be the interior of $\bar{\square}_j$, where $\bar{\square}_j$ is the compact convex polygon in $\mathbb{R}^2 = \{(u, v)\}$ defined by

$$\bar{\square}_j = \begin{cases} \{(u, v) \in \mathbb{R}^2; u + v \leq 1, u \geq -1, v \geq -1\}, & j = 1, \\ \{(u, v) \in \mathbb{R}^2; |u| \leq 1, |v| \leq 1\}, & j = 2, \\ \{(u, v) \in \mathbb{R}^2; |u + v| \leq 1, |u| \leq 1, |v| \leq 1\}, & j = 3. \end{cases}$$

Put $u := h_x = \partial h / \partial x$ and $v := h_y = \partial h / \partial y$. Since the moment map sending each $(x, y) \in \mathbb{R}^2$ to $(u(x, y), v(x, y)) \in \square_j$ defines a diffeomorphism between \mathbb{R}^2 and \square_j , every function on $\mathbb{R}^2 = \{(x, y)\}$ is naturally regarded as a function on $\square_j = \{(u, v)\}$ via this moment map, and vice versa. We now consider the Legendre transform $h^* := xu + yv - h$. Then $h^* = h^*(u, v)$ regarded as a function in u and v satisfies

$$x = h_u^* \quad \text{and} \quad y = h_v^*,$$

where $h_u^* := \partial h^* / \partial u$ and $h_v^* := \partial h^* / \partial v$. Then by (2.1) and

$$(2.2) \quad \begin{pmatrix} h_{uu}^* & h_{uv}^* \\ h_{uv}^* & h_{vv}^* \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}^{-1} = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix}^{-1},$$

we have $e^h h_{xx} = h_{vv}^*$, $e^h h_{yy} = h_{uu}^*$ and $e^h h_{xy} = -h_{uv}^*$. Now by $h_{vvu}^* = h_{uvv}^*$ and $h_{uuv}^* = h_{uvu}^*$, we see that

$$(2.3) \quad \frac{\partial(e^h h_{xx})}{\partial u} = \frac{\partial(-e^h h_{xy})}{\partial v} \quad \text{and} \quad \frac{\partial(e^h h_{yy})}{\partial v} = \frac{\partial(-e^h h_{xy})}{\partial u}.$$

From the first equality of (2.3), we obtain

$$\frac{\partial h}{\partial u} h_{xx} + \frac{\partial h}{\partial v} h_{xy} + \frac{\partial h_{xx}}{\partial u} + \frac{\partial h_{xy}}{\partial v} = 0.$$

Hence, together with $\frac{\partial h}{\partial u} = \frac{\partial x}{\partial u} h_x + \frac{\partial y}{\partial u} h_y$ and $\frac{\partial h}{\partial v} = \frac{\partial x}{\partial v} h_x + \frac{\partial y}{\partial v} h_y$, it now follows that

$$(h_{uu}^* u + h_{uv}^* v) h_{xx} + (h_{uv}^* u + h_{vv}^* v) h_{xy} + \frac{\partial h_{xx}}{\partial u} + \frac{\partial h_{xy}}{\partial v} = 0,$$

where we used (2.2) and the definitions of u and v . Again by (2.2), $h_{uu}^* h_{xx} + h_{uv}^* h_{xy} = 1$ and $h_{uv}^* h_{xx} + h_{vv}^* h_{xy} = 0$. Then

$$u + \frac{\partial h_{xx}}{\partial u} + \frac{\partial h_{xy}}{\partial v} = 0.$$

Hence we obtain

$$(2.4) \quad \frac{\partial(h_{xx} + \frac{1}{3}u^2)}{\partial u} = \frac{\partial(-h_{xy} - \frac{1}{3}uv)}{\partial v}.$$

Similarly, from the second equality of (2.3), we obtain

$$(2.5) \quad \frac{\partial(h_{yy} + \frac{1}{3}v^2)}{\partial v} = \frac{\partial(-h_{xy} - \frac{1}{3}uv)}{\partial u}.$$

Let $0 < \varepsilon \ll 1$. For each $p \in \mathbb{R}^2 = \{(u, v)\}$, let $U_\varepsilon(p)$ denote the ε -neighborhood of p in \mathbb{R}^2 . We now put

$$(\square_j)_\varepsilon := \bigcup_{p \in \square_j} U_\varepsilon(p).$$

Note that (2.4) and (2.5) hold on \square_j . To see whether (2.4) and (2.5) are true also for $(\square_j)_\varepsilon$, take an arbitrary point q in $X_j \setminus T$. If necessary, replace the complex coordinates (z_1, z_2) for $T = (\mathbb{C}^*)^2$ by

$$(z_1^{\alpha_1} z_2^{\beta_1}, z_1^{\alpha_2} z_2^{\beta_2}) \quad \text{for some} \quad \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \in \text{GL}(2, \mathbb{Z}).$$

Then we may assume that z_1, z_2 regarded as meromorphic functions on X_j are holomorphic at q satisfying

$$z_1(q) = 0.$$

Put $b := |z_2(q)|^2 \geq 0$. Note that there is a real-analytic function $Q = Q(r_1, r_2)$ in two variables r_1, r_2 defined in a neighborhood of $(0, b)$ such that $e^{-h} = Q(|z_1|^2, |z_2|^2)$. Then

$$h_x = -r_1 Q^{-1} \frac{\partial Q}{\partial r_1} \quad \text{and} \quad h_y = -r_1 Q^{-1} \frac{\partial Q}{\partial r_2},$$

where both are evaluated at $(r_1, r_2) = (|z_1|^2, |z_2|^2)$. Then by allowing r_1 (and r_2 as well if $b = 0$) to take negative values (see also [9;p.722-723]), we see that the terms h_{xx} , h_{xy} , h_{yy} are well-defined also on $(\square_j)_\varepsilon \setminus \square_j$. Then both (2.4) and (2.5) hold on $(\square_j)_\varepsilon$. Since $(\square_j)_\varepsilon$ is simply connected, we now obtain real-analytic functions $H_1 = H_1(u, v)$ and $H_2 = H_2(u, v)$ on $(\square_j)_\varepsilon$ such that

$$(2.6) \quad \frac{\partial H_1}{\partial v} = h_{xx} + \frac{1}{3}u^2 \quad \text{and} \quad \frac{\partial H_1}{\partial u} = -h_{xy} - \frac{1}{3}uv;$$

$$(2.7) \quad \frac{\partial H_2}{\partial u} = h_{yy} + \frac{1}{3}v^2 \quad \text{and} \quad \frac{\partial H_2}{\partial v} = -h_{xy} - \frac{1}{3}uv.$$

In view of the second equalities of (2.6) and (2.7), we obtain $\frac{\partial H_1}{\partial u} = \frac{\partial H_2}{\partial v}$. Hence, for some real-analytic function $H = H(u, v)$ on $(\square_j)_\varepsilon$,

$$(2.8) \quad H_1 = \frac{\partial H}{\partial v} \quad \text{and} \quad H_2 = \frac{\partial H}{\partial u}.$$

By (2.8) together with the first equalities of (2.6) and (2.7), we obtain the following on $(\square_j)_\varepsilon$:

$$(2.9) \quad h_{xx} = H_{vv} - \frac{1}{3}u^2, \quad h_{yy} = H_{uu} - \frac{1}{3}v^2, \quad h_{xy} = -H_{uv} - \frac{1}{3}uv,$$

where $H_{uu} := (\partial^2 H)/(\partial u^2)$, $H_{uv} := (\partial^2 H)/(\partial u \partial v)$, $H_{vv} := (\partial^2 H)/(\partial v^2)$. On the other hand, by (2.2) and (2.9),

$$\begin{aligned} \frac{\partial e^{-h}}{\partial u} &= -e^{-h} \left(h_x \frac{\partial x}{\partial u} + h_y \frac{\partial y}{\partial u} \right) = -e^{-h} (u h_{uu}^* + v h_{uv}^*) \\ &= -u h_{yy} + v h_{xy} = -u \left(H_{uu} - \frac{1}{3}v^2 \right) + v \left(-H_{uv} - \frac{1}{3}uv \right) \\ &= -(u H_{uu} + v H_{uv}) = \frac{\partial}{\partial u} (H - u H_u - v H_v). \end{aligned}$$

Similarly,

$$\frac{\partial e^{-h}}{\partial v} = \frac{\partial}{\partial v} (H - u H_u - v H_v).$$

Hence $e^{-h} = H - u H_u - v H_v + C$ for some real constant C . Replacing H by $H + C$, we may assume without loss of generality that

$$(2.10) \quad e^{-h} = H - u H_u - v H_v.$$

In view of (2.9) and (2.10), the equation (2.1) is rewritten as

$$(2.11) \quad \begin{vmatrix} H_{vv} - \frac{1}{3}u^2, & H_{uv} + \frac{1}{3}uv \\ H_{uv} + \frac{1}{3}uv, & H_{uu} - \frac{1}{3}v^2 \end{vmatrix} = H - uH_u - vH_v.$$

Put $s := H_u$ and $t := H_v$. In the next section, we shall show that the image of $\bar{\square}_j$ under the mapping

$$\bar{\square}_j \ni (u, v) \mapsto (s(u, v), t(u, v)) \in \mathbb{R}^2$$

is nothing but $\bar{\Omega}_j$ in the introduction. Moreover, by this map, the boundary $\partial\bar{\square}_j$ is mapped onto the boundary $\partial\Omega_j$. We now consider the Legendre transform $\psi := uH_u + vH_v - H$. Regard ψ as a function in $(s, t) \in \bar{\Omega}_j$. Since $\text{Hess}(h)$ is positive on T and vanishes on $X_j \setminus T$, we see from (2.11) that ψ is negative on Ω_j , and vanishes just on the boundary $\partial\Omega_j$. Then

$$(2.12) \quad \psi_s = u \quad \text{and} \quad \psi_t = v.$$

Moreover, in view of the equalities $\psi_{ss} = \partial u / \partial s$, $\psi_{tt} = \partial v / \partial t$ and $\psi_{st} = \partial u / \partial t$, we see that

$$(2.13) \quad \begin{pmatrix} \psi_{ss} & \psi_{st} \\ \psi_{st} & \psi_{tt} \end{pmatrix} = \begin{pmatrix} H_{uu} & H_{uv} \\ H_{uv} & H_{vv} \end{pmatrix}^{-1}.$$

In particular, $\det \text{Hess}(H) := H_{uu}H_{vv} - (H_{uv})^2$ and $\det \text{Hess}(\psi) := \psi_{ss}\psi_{tt} - (\psi_{st})^2$ satisfy $\det \text{Hess}(H) \cdot \det \text{Hess}(\psi) = 1$. Now by (2.11),

$$(2.14) \quad \det \text{Hess}(H) = -\psi + \frac{1}{3}(u^2H_{uu} + 2uvH_{uv} + v^2H_{vv}).$$

By (2.13), we have $H_{uu} = \psi_{tt} / \det \text{Hess}(\psi)$, $H_{vv} = \psi_{ss} / \det \text{Hess}(\psi)$, $H_{uv} = -\psi_{st} / \det \text{Hess}(\psi)$. Hence, from (2.12) and (2.14), it follows that

$$\begin{aligned} 1 &= \det \text{Hess}(\psi) \left\{ -\psi + \frac{1}{3}(u^2H_{uu} + 2uvH_{uv} + v^2H_{vv}) \right\} \\ &= -\psi \det \text{Hess}(\psi) + \frac{1}{3} \{ \psi_s^2 \psi_{tt} - 2\psi_s \psi_t \psi_{st} + \psi_t^2 \psi_{ss} \}. \end{aligned}$$

Thus, we obtain the equality (1.3). By setting (1.2), we finally see that (1.1) holds, as required.

3. THE BOUDARY CONDITION

We first consider the case $j = 1$, so that $X_j = \mathbb{P}^2(\mathbb{C})$. Then the Kähler-Einstein form ω on X_j given by

$$h = -\log 9 - x - y + 3 \log(1 + e^x + e^y).$$

is known as the Fubini-Study form. This obviously satisfies the equation (2.1). Moreover, $u := h_x$ and $v := h_y$ satisfy the inequalities

$$\begin{aligned} 1 - (u + v) &= \frac{3}{1 + e^x + e^y} \geq 0, \\ u + 1 &= \frac{3e^x}{1 + e^x + e^y} \geq 0, \quad v + 1 = \frac{3e^y}{1 + e^x + e^y} \geq 0. \end{aligned}$$

In this case, H and ψ are

$$\begin{aligned} H &= \frac{uv(u+v)}{6} + \frac{u^2 + uv + v^2}{3} + \frac{1}{3}, \\ \psi &= uH_u + vH_v - H = \frac{1}{3}(u+1)(v+1)(u+v-1) \leq 0. \end{aligned}$$

Then h and H satisfy (2.9) and (2.11). Moreover ψ , when regarded as a function on $\bar{\square}_j$, is negative on \square_j vanishing on the boundary $\partial\square_j$. In addition to this, for $s := H_u$ and $t := H_v$, we can easily check that

$$\begin{cases} \rho_{11}(s, t) = 0 & \text{on the line } u + v = 1, \\ \rho_{21}(s, t) = 0 & \text{on the line } u = -1, \\ \rho_{31}(s, t) = 0 & \text{on the line } v = -1, \end{cases}$$

and that the mapping $\bar{\square}_j \ni (u, v) \mapsto (s(u, v), t(u, v)) \in \bar{\Omega}_j$ takes \square_j diffeomorphically onto Ω_j . We now regard ψ as a function on $\bar{\Omega}_j$. Then ψ is negative on Ω_j vanishing on the boundary $\partial\Omega_j$. We also see that ψ is a root of a polynomial of degree 4 with coefficients in $\mathbb{Q}[s, t]$ such that the leading coefficient is 1. Moreover, the asymptotic expansion of ψ along the boundary curve $\partial\Omega_j$, especially along $\{u = -1\} \cap \partial\square_j$, shows

$$\psi = -\rho_{21} + \frac{(\rho_{21})^2}{-4s + 2t - 1} + \text{higher order terms in } \rho_{21},$$

where from this expression of $P(\Omega_j)$, we easily see that the pullback of the Kähler-Einstein form ω to each (irreducible) component of $X_j \setminus T$ is nothing but the Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$.

We next consider the case $j = 2$, so that $X_j = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Then we fix the Kähler-Einstein form ω on X_j defined by

$$h = -2 \log 2 - x - y + 2 \log(1 + e^x) + 2 \log(1 + e^y).$$

This again satisfies the equation (2.1). Then $u := h_x$ and $v := h_y$ satisfies

$$\begin{aligned} -1 &\leq -1 + \frac{2}{1 + e^x} = u = 1 - \frac{2e^x}{1 + e^x} \leq 1, \\ -1 &\leq -1 + \frac{2}{1 + e^y} = v = 1 - \frac{2e^y}{1 + e^y} \leq 1. \end{aligned}$$

In this case, H and ψ are expressible as

$$\begin{aligned} H &= \frac{u^2 + v^2}{4} - \frac{u^2 v^2}{12} + \frac{1}{4}, \\ \psi &= -\frac{1}{4}(1 - u^2)(1 - v^2) \leq 0. \end{aligned}$$

Then h and H satisfy (2.9) and (2.11). Moreover ψ , when regarded as a function on $\bar{\square}_j$, is negative on \square_j vanishing on the boundary $\partial \square_j$. Now for $s := H_u$ and $t := H_v$, we see that

$$\left\{ \begin{array}{ll} \rho_{12}(s, t) = 0 & \text{on the line } v = 1, \\ \rho_{22}(s, t) = 0 & \text{on the line } u = 1, \\ \rho_{32}(s, t) = 0 & \text{on the line } v = -1, \\ \rho_{42}(s, t) = 0 & \text{on the line } u = -1, \end{array} \right.$$

and the mapping $\bar{\square}_j \ni (u, v) \mapsto (s(u, v), t(u, v)) \in \bar{\Omega}_j$ takes \square_j diffeomorphically onto Ω_j . Then ψ regarded as a function on $\bar{\Omega}_j$ is negative on Ω_j and vanishes on the boundary $\partial \Omega_j$. We also see that ψ is a root of a polynomial of degree 5 with coefficients in $\mathbb{Q}[s, t]$ such that the leading coefficient is 1. The asymptotic expansion of ψ along the boundary curve is, when restricted to a neighborhood of $\{u = 1\} \cap \partial \square_j$ for instance,

$$\psi = -\rho_{22} + \frac{(\rho_{22})^2}{6s - 2} + \text{higher order terms in } \rho_{22}.$$

In view of this expression of $P(\Omega_j)$, it again follows that the pullback of the Kähler-Einstein form ω to each component of $X_j \setminus T$ is the Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$.

We finally consider the case $j = 3$, so that $X_j = \mathbb{P}^2(\mathbb{C}) \# 3\bar{\mathbb{P}}^2(\mathbb{C})$. Then we may assume without loss of generality that the K -invariant Kähler-Einstein metric ω on X_j is invariant under the natural action of the subgroup $D_6 \subset \text{Aut}(X_j)$, where D_6 denotes the dihedral group of order 12. Hence from the degeneracy condition for the matrix $\text{Hess}(h)$ along $X_j \setminus T$, we now see that

$$H|_{\partial\Box_j} = \frac{1}{6}(r+1),$$

where $r := u^2 + uv + v^2$. Put $\tau := (1 - u^2)(1 - v^2)(1 - (u + v)^2)$. Then the power series expansion of H along the boundary $\partial\Box_j$ is given by

$$(3.1) \quad H = \frac{1}{6}(r+1) + \frac{\tau}{12(r-3)} + \sum_{\alpha \geq 2} \eta_\alpha \tau^\alpha,$$

where each coefficient $\eta_\alpha = \eta_\alpha(r)$ ($\alpha \geq 3$) is defined inductively from $\eta_2, \eta_3, \dots, \eta_{\alpha-1}$ (and their derivatives). Now by this (3.1), it is easily seen that $s := H_u(u, v)$ and $t := H_v(u, v)$ satisfy

$$\left\{ \begin{array}{ll} \rho_{13}(s, t) = 0 & \text{on } \{u + v = 1\} \cap \partial\Box_j, \\ \rho_{23}(s, t) = 0 & \text{on } \{u = 1\} \cap \partial\Box_j, \\ \rho_{33}(s, t) = 0 & \text{on } \{v = -1\} \cap \partial\Box_j, \\ \rho_{43}(s, t) = 0 & \text{on } \{u + v = -1\} \cap \partial\Box_j, \\ \rho_{53}(s, t) = 0 & \text{on } \{u = -1\} \cap \partial\Box_j, \\ \rho_{63}(s, t) = 0 & \text{on } \{v = 1\} \cap \partial\Box_j, \end{array} \right.$$

where the map $\bar{\Box}_j \ni (u, v) \mapsto (s(u, v), t(u, v)) \in \bar{\Omega}_j$ again takes \Box_j diffeomorphically onto Ω_j . Note that the term η_2 is uniquely determined by the “Fubini-Pick invariant” $P(\Omega_j)$, and vice versa. Then by (3.1), we can explicitly describe h (and hence ω) from the data $P(\Omega_j)$, since the equalities (2.2), (2.9), (2.10) above allow us to recover h from H by

$$\begin{aligned} x &= \int \frac{(H_{uu} - \frac{1}{3}v^2)du + (H_{uv} + \frac{1}{3}uv)dv}{H - uH_u - vH_v}, \\ y &= \int \frac{(H_{vv} - \frac{1}{3}u^2)dv + (H_{uv} + \frac{1}{3}uv)du}{H - uH_u - vH_v}, \\ h &= -\ln(H - uH_u - vH_v). \end{aligned}$$

We also see that η_2 (and hence $P(\Omega_j)$) is uniquely determined by the pullback of the Kähler-Einstein form ω to $X_j \setminus T$, and vice versa. In particular, it is seen that the pullback of the Kähler form ω to any irreducible components of $X_j \setminus T$ can never be a Kähler-Einstein form. The details in this case $j = 3$ and also in the case $X = \mathbb{P}^2(\mathbb{C}) \# 2\bar{\mathbb{P}}^2(\mathbb{C})$ (cf. §4) will be published elsewhere.

4. CONCLUDING REMARKS

The Kähler-Ricci soliton on the toric surface $\mathbb{P}^2(\mathbb{C}) \# \bar{\mathbb{P}}^2(\mathbb{C})$ is explicitly written (cf. [7]) by solving an ODE. Now the remaining toric surface is $X := \mathbb{P}^2(\mathbb{C}) \# 2\bar{\mathbb{P}}^2(\mathbb{C})$. As in the preceding sections, the equation for a K -invariant Kähler-Ricci soliton on X (cf. [16]) is

$$(4.1) \quad \det \text{Hess}(h) = e^{-h - \alpha(u+v)},$$

where $0 \neq \alpha \in \mathbb{R}$ is such that $\alpha(u+v)$ is the Hamiltonian function for the holomorphic vector field (cf. [15]) associated to the Kähler-Ricci soliton. Let $\square := \bar{\square} \setminus \partial\square$ be the interior of the polygon $\bar{\square}$ in \mathbb{R}^2 defined by

$$\bar{\square} = \{(u, v) \in \mathbb{R}^2; |u| \leq 1, |v| \leq 1, u + v \leq 1\}.$$

Then by the same argument as in obtaining (2.11) from (2.1), we can reduce (4.1) to the following equation in $H \in C^\omega(\bar{\square})_{\mathbb{R}}$ with a suitable boundary condition:

$$\begin{aligned} & \left| \begin{array}{cc} H_{vv} + 2\alpha H_v + \alpha^2 H - \frac{1}{\alpha}u, & H_{uv} + \alpha H_v + \alpha H_u + \alpha^2 H - \frac{1}{\alpha^2} \\ H_{uv} + \alpha H_v + \alpha H_u + \alpha^2 H - \frac{1}{\alpha^2}, & H_{uu} + 2\alpha H_u + \alpha^2 H - \frac{1}{\alpha}v \end{array} \right| \\ &= H - u(H_u + \alpha H) - v(H_v + \alpha H) + \frac{uv}{\alpha^2}. \end{aligned}$$

Since $\partial\square$ is a 1-cycle, we have the following compatibility condition for the boundary values of H_u and H_v :

$$(2 - \alpha^2)e^{3\alpha} = 4e^{2\alpha} - 2(1 + \alpha),$$

where α is characterized as the nonzero solution of this equation. Then this fits to the approximate value of the constant α in [6; (14)] (see also [15; Lemma 2.2]). However, in this Kähler-Ricci soliton case, the equation cannot be so simplified as in (1.1) and (1.3).

As compared with [5] and [6], the results in this note may give another frame work for numerical studies of Kähler-Einstein metrics and Kähler-Ricci solitons. Let me finally remark that parts of this note are in [10], and were announced in Aug., 1987 in the Taniguchi International Symposium at Katata.

REFERENCES

- [1] E. Calabi, Complete affine hyperspheres. I, *Symp. Math.* 10 (1972), 19–38.
- [2] S.-Y. Chen & S.-T. Yau, On the regularity of the Monge-Ampere Equation $\det(\partial^2 u / \partial x_i \partial x_j) = F(x, u)$, *Comm. Pure Appl. Math.*, 30 (1977), 41–68.
- [3] S.-Y. Chen & S.-T. Yau, Complete affine hypersurfaces. Part I. The completeness of affine metrics, *Comm. Pure Appl. Math.*, 39 (1986), 839–866.
- [4] S.K. Donaldson, Scalar curvature and stability of toric varieties, *J. Differential Geom.*, 62 (2002), 289–349.
- [5] C. Doran, M. Headrick, C.P. Herzog, J. Kantor & T. Wiseman, *Numerical Kähler-Einstein metric on the third del Pezzo*, arXiv:hep-th/0703057v2.
- [6] M. Headrick & T. Wiseman, *Numerical Kähler-Ricci soliton on the second del Pezzo*, arXiv:math.DG/0706.2329v1.
- [7] N. Koiso, On rotationally symmetric Hamilton’s equation for Kähler-Einstein metrics, *Adv. Stud. Pure Math.*, Academic Press & Kinokuniya, 18-I (1990), 327–337.
- [8] J.C. Loftin, Affine spheres and Kähler-Einstein metrics, *Math. Res. Lett.*, 9 (2002), 425–432.
- [9] T. Mabuchi, Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties, *Osaka J. Math.*, 24 (1987), 705–737.
- [10] T. Mabuchi, *Toric domains*, notes in Japanese (unpublished), 1986.
- [11] T. Sasaki, *On the characteristic function of a strictly convex domain and the Fubini-Pick invariant*, preprint series, Max-Planck-Inst. für Math., Bonn, 1987.
- [12] Y.-T. Siu, The existence of Kähler-Einstein metrics with positive anticanonical line bundle and a suitable finite symmetry group, *Ann. of Math.*, 127 (1988), 585–627.
- [13] G. Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$, *Invent. Math.*, 89 (1987), 225–246.
- [14] G. Tian & S.-T. Yau, On Kähler-Einstein metrics on complex surfaces with $c_1 > 0$, *Commun. Math. Phys.*, 112 (1987), 175–203.
- [15] G. Tian & X. Zhu, A new holomorphic invariant and uniqueness of Kähler-Ricci solitons, *Comment. Math. Helv.*, 77 (2002), 297–325.
- [16] X.-J. Wang & X. Zhu, Kähler-Ricci solitons on toric manifolds with positive first Chern class, *Adv. in Math.*, 188 (2004), 87–103.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA
UNIVERSITY, TOYONAKA, OSAKA, 560-0043 JAPAN